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The Effect of the Reference Frame on the Thermophysical Properties of an Ideal Gas[1]

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The effect that the frame of reference has on the thermophysical properties of an ideal gas is examined from a fundamental theoretical standpoint based on the Boltzmann equation. In continuum mechanics, the principle of material frame-indifference forbids the thermophysical properties of a fluid or solid to depend in any way on the motion of the reference frame. It is demonstrated that the Boltzmann equation is only consistent with material frame-indifference in a strong approximate sense provided that the gas is not highly rarefied and, thus, well within the limits of classical continuum mechanics. Estimates of the mean free times for which material frame-indifference can be invoked in the modeling of gas flows are provided from an analysis of the problem of heat conduction in a rigidly rotating gas. Applications of these results in obtaining asymptotic solutions of the Boltzmann equation for the continuum description of an ideal gas are discussed briefly.

KEY WORDS: Boltzmann equation; continuum mechanics; frame-indifference; kinetic theory; mean free time.

1. INTRODUCTION

A controversy has developed during the past decade over the consistency of the principle of material frame-indifference of modern continuum mechanics with the kinetic theory of gases [1–11]. Material frame-indif-

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ference requires constitutive equations for the stress and heat flux to be form invariant under arbitrary time-dependent rotations and translations of the frame of reference. However, expressions obtained by the Chapman–Enskog and Maxwellian iterations for an ideal gas (which are analogous to constitutive equations in continuum mechanics) depend explicitly on the rotation rate of the framing through the spin tensor, in apparent violation of this principle [1, 2]. This disparity in invariance properties has created two opposing camps among researchers in this field: those who believe that the iterative procedures for solving the Boltzmann equation are defective and those who believe that material frame-indifference should be abandoned as a general principle of continuum mechanics.

The purpose of the present paper is to review the most recent developments in this controversy and to present an alternative position which, as it turns out, lies somewhere in between the two extreme opposing views cited above. Unlike in the previous studies [1–11], this question is approached by a direct analysis of the Boltzmann equation. The main physical consequence of material frame-indifference (namely, the fact that it forbids the *values* of the stress tensor and heat flux to be altered by the superposition of a rigid body motion on a given thermomechanical process) is compared with the Boltzmann equation for the test problem of heat conduction in a rigidly rotating gas. In this manner, the problem is simplified to the examination of values of constitutive functions rather than their invariance properties, thus making a direct comparison with the Boltzmann equation possible (a preliminary investigation along these lines was presented by Speziale [12]). It is demonstrated that the Boltzmann equation, in its full generality, is *not* consistent with the main physical consequence of material frame-indifference mentioned above as a result of unbalanced molecular Coriolis forces. However, the two are in excellent approximate agreement provided that the mean free time of the gas is extremely small. Since constitutive equations, in the classical continuum mechanics sense, require equivalently small mean free times for their existence, it is argued that the Boltzmann equation supports the application of material frame-indifference for classical continuum mechanics problems. However, it does break down for highly rarefied gases, where the thermophysical properties can vary depending on whether the frame of reference is inertial or not. Estimates of the mean free times for which this occurs are provided, along with possible applications of these results to the modeling of transport processes in ideal gases.

2. THE CONTROVERSY OVER MATERIAL FRAME-INDIFFERENCE

In classical continuum mechanics, the balance laws are those for mass, linear momentum, angular momentum, and energy which, respectively, take the form

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_k}(\rho v_k)=0 \tag{1}$$

$$\rho \dot{v}_k=\frac{\partial T_{kl}}{\partial x_l}+\rho b_k \tag{2}$$

$$T_{kl}=T_{lk} \tag{3}$$

$$\rho \dot{\varepsilon}=T_{kl}\frac{\partial v_l}{\partial x_k}-\frac{\partial q_k}{\partial x_k} \tag{4}$$

(cf., Truesdell and Noll [13]), where ρ is the mass density, $\mathbf{v}$ is the velocity field, $\mathbf{T}$ is the stress tensor, $\mathbf{b}$ is the external body force per unit mass, ε is the internal energy density, and $\mathbf{q}$ is the heat flux vector. In Eqs. (1)–(4), the Einstein summation convention applies to repeated indices, a superposed dot denotes the material time derivative, and the presence of any energy sources has been neglected. The equations of motion (1)–(4) are not closed unless constitutive equations are provided that tie $\mathbf{T}$, $\mathbf{q}$, and ε to the global history of the motion and temperature of the continuum. For fluids, these constitutive equations are of the general form

$$\begin{gathered}\mathbf{T}(\mathbf{x}, t)=\mathbf{T}[\mathbf{v}(\mathbf{x}', t'), \theta(\mathbf{x}', t'); \mathbf{x}, t]\\ \mathbf{x}'\in\mathscr{V}, \qquad t'\in(-\infty, t)\end{gathered} \tag{5}$$

$$\begin{gathered}\mathbf{q}(\mathbf{x}, t)=\mathbf{q}[\mathbf{v}(\mathbf{x}', t'), \theta(\mathbf{x}', t'); \mathbf{x}, t]\\ \mathbf{x}'\in\mathscr{V}, \qquad t'\in(-\infty, t)\end{gathered} \tag{6}$$

$$\begin{gathered}\varepsilon(\mathbf{x}, t)=\varepsilon[\mathbf{v}(\mathbf{x}', t'), \theta(\mathbf{x}', t'); \mathbf{x}, t]\\ \mathbf{x}'\in\mathscr{V}, \qquad t'\in(-\infty, t)\end{gathered} \tag{7}$$

where θ is the absolute temperature and the bracket denotes a functional (i.e., any quantity determined by a function). The principle of material frame-indifference of modern continuum mechanics requires that these constitutive equations be form invariant under arbitrary time-dependent rotations and translations of the spatial frame of reference [13]. Consequently, when material frame-indifference is invoked, the constitutive Eqs. (5)–(7) will be of the same form whether or not the frame is inertial.

The kinetic theory for a dilute monatomic gas is based on the Boltzmann equation, which takes the form

$$\frac{\partial f}{\partial t}+\mathbf{c}\cdot\nabla f+\mathbf{b}\cdot\frac{\partial f}{\partial \mathbf{c}}=\mathbb{C}[f] \tag{8}$$

where f is the phase density function, $\mathbf{c}$ is the molecular velocity, and $\mathbb{C}[f]$ is the collision operator (cf. Chapman and Cowling [14]), which is not written out in detail since the results obtained will not depend on its specific form. The molecular velocity $\mathbf{c}$ is decomposed into a macroscopic and peculiar velocity, respectively, as follows [14]

$$\mathbf{c}=\mathbf{v}+\mathbf{u} \tag{9}$$

where

$$\mathbf{v}=\frac{1}{n}\int_{S_c}\mathbf{c}f\,d^3c \tag{10}$$

and n is the number density, i.e.,

$$n=\int_{S_c} f\,d^3c \tag{11}$$

Various other macroscopic fields are then defined as follows:

$$\rho=mn, \qquad \theta=\frac{m}{3kn}\int_{S_c}\mathbf{u}\cdot\mathbf{u}f\,d^3c \tag{12}$$

$$T_{kl}=-m\int_{S_c}u_k u_l f\,d^3c, \qquad q_k=\frac{m}{2}\int_{S_c}\mathbf{u}\cdot\mathbf{u}u_k f\,d^3c \tag{13}$$

where m is the molecular mass and k is the Boltzmann constant. The kinetic theory gives rise to the expression [14]

$$\varepsilon=\frac{3}{2}\frac{k}{m}\theta \tag{14}$$

which is a special case of the more general thermomechanical constitutive Eq. (7). Furthermore, by taking moments of the Boltzmann equation with m, $\mathbf{c}$, and $\mathbf{c}\cdot\mathbf{c}$, respectively, the balance laws of continuum mechanics given by Eqs. (1), (2), and (4) are obtained. The symmetry of the stress tensor follows from its definition given in Eq. (13). Hence, the kinetic theory of gases is comparable to continuum mechanics if expressions for the stress tensor and heat flux of the form of Eqs. (5) and (6) can be obtained from the Boltzmann equation. Provided that the gas is not highly rarefied, the Chapman–Enskog iteration, the Maxwellian iteration, or a generalized mean free path model can be used to obtain higher-order approximations for the stress tensor and heat flux vector that transcend Navier–Stokes theory and Fourier's law of heat conduction. The expressions obtained from these approaches are of the same general form as the continuum constitutive Eqs. (5) and (6). However, they contain the spin tensor

$$\omega_{kl}=\frac{1}{2}\left(\frac{\partial v_k}{\partial x_l}-\frac{\partial v_l}{\partial x_k}\right) \tag{15}$$

which is a frame-dependent tensor. More specifically, the spin tensor in a rotating frame of reference $\mathbf{x}^*$ is related to that in an inertial framing $\mathbf{x}$ by the equation

$$\boldsymbol{\omega}=\boldsymbol{\omega}^*+\mathrm{dual}\,\boldsymbol{\Omega} \tag{16}$$

where dual $\boldsymbol{\Omega}$ is the antisymmetric tensor formed from the rotation rate $\boldsymbol{\Omega}$ of the framing $\mathbf{x}^*$. Consequently, these iterative procedures yield expressions for the stress tensor and heat flux vector in an ideal gas, which depend on the motion of the framing in violation of material frame-indifference.

Müller [1] and Edelen and McLennan [2] demonstrated that the Maxwellian iteration and Chapman–Enskog iteration, respectively, give rise to expressions for the stress and heat flux which are frame-dependent starting with the third approximation (i.e., the Burnett equations). These authors then argued that the kinetic theory of gases was not in support of material frame-indifference, and hence, constitutive equations should be allowed to depend on the frame of reference. Subsequently, Söderholm [3] claimed to produce a simple physical argument that justified the presence of the frame-dependent terms in the Burnett equations. This led him to conclude that there were severe limitations on the domain of validity of material frame-indifference. Wang [4] took exception to the claims of Müller [1] and Edelen and McLennan [2] on the grounds that these iterative schemes, at best, yield approximate expansions for the stress and heat flux and it is well known that such approximations can destroy invariance properties. He gave the counterexample of the Taylor expansion for the $\sin x$ given by

$$\sin x=x-\frac{x^3}{3!}+\cdots \tag{17}$$

While the $\sin x$ is periodic (i.e., invariant under shifts of 2π in the variable x), no partial sum of its expansion exhibits the same invariance property. Truesdell [5] argued that while the results obtained by the Chapman–Enskog and Maxwellian iterations are similar in form to constitutive equations in continuum mechanics, it is wrong to regard them as such since the variables entering these equations are not independent. He then proceeded to argue that it is probably not possible even to ask the question as to whether these kinetic theory results are consistent with material frame-indifference since a time-dependent rotation could give rise to a velocity field which is inconsistent with the Boltzmann equation, thus making the equations obtained by these iterative procedures invalid when subjected to a change of frame. Speziale [6] explicitly showed where the frame dependence enters into the Chapman–Enskog iteration and demonstrated that if a given macroscopic process is a member of the Chapman–Enskog class, then one which differs by an arbitrary rigid body motion will not be. The author than argued that the iterative procedures in the kinetic theory are defective and should be abandoned in favor of ones which are consistent with material frame-indifference.

Woods [7], who had previously shown that the same kind of frame-dependent terms which appear in the Burnett equations can be obtained from a generalized mean free path model, staunchly argued that material frame-indifference is valid only for the linear theory and should be abandoned as a general axiom of continuum mechanics. This initiated a debate with Green [8, 9], who took the opposite position in support of material frame-indifference. Woods [10] eventually reversed his position and developed a frame-indifferent kinetic theory by altering the definition of the peculiar velocity. More recently, Murdoch [11] argued that the criticisms of material frame-indifference based on these iterative procedures in the kinetic theory are without foundation. He claimed that there is actually no conflict with material frame-indifference since the equations obtained from these iterative procedures only depend on the rotation rate $\mathbf{\Omega}$ through the intrinsic spin tensor

$$\boldsymbol{\omega}^* + \text{dual}\,\mathbf{\Omega}$$

This is the spin tensor relative to an inertial framing—an object which he argued is a valid constitutive variable since it allows for the *values* of the stress and heat flux obtained from it to transform in an invariant manner under a change of observer. However, it should be noted that such constitutive equations are *not* consistent with the principle of material frame-indifference as it is usually formulated [13], since in order to write the correct form of the equation it is necessary to know a priori the motion of the given frame of reference relative to an inertial one.

The conflicting arguments over material frame-indifference, briefly reviewed above, have generated a considerable amount of confusion among researchers in mechanics. In the next section, an attempt is made to clarify this issue by a direct appeal to the Boltzmann equation. A preliminary investigation along these lines was presented by Speziale [12] which yielded conclusions that lie somewhere in between the extreme opposing positions cited above (including those initially proposed by the author [6]).

3. THE BOLTZMANN EQUATION AND HEAT CONDUCTION IN A RIGIDLY ROTATING GAS

The general problem to be considered is that of steady heat conduction in a rigidly rotating gas (with constant angular velocity $\mathbf{\Omega}$) that can be initiated by the application of boundary conditions which have a disparity in temperature. An extra body force of the amount

$$\mathbf{b}_E = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}^*) \tag{18}$$

will be applied in the rotating frame $\mathbf{x}^*$ in order to balance the centrifugal acceleration generated by the rotation. This physical problem was chosen since it contains the critical effect of rigid rotations in a relatively simple context. It was proven by Speziale [12] that the Boltzmann equation in the kinetic theory is completely consistent with material frame-indifference insofar as translational accelerations of the frame of reference are concerned (Wang [4] first noted that the results of the Chapman–Enskog and Maxwellian iterations were invariant under translational accelerations of the framing). Another important reason for choosing this problem is that it was considered by Müller [1] and Söderholm [3] for the Burnett approximation. In this way, a basis for comparison is established.

The field Eqs. (1), (2), and (4) of continuum mechanics take the coordinate free from [1]

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \mathbf{v}^*) = 0 \tag{19}$$

$$\rho^* \dot{\mathbf{v}}^* = \nabla^* \cdot \mathbf{T}^* + \rho^* \mathbf{b}^* - \rho^* \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}^*) - 2\rho^* \mathbf{\Omega} \times \mathbf{v}^* \tag{20}$$

$$\rho^* \dot{\varepsilon}^* = \mathrm{tr}(\mathbf{T}^* \cdot \nabla^* \mathbf{v}^*) - \nabla^* \cdot \mathbf{q}^* \tag{21}$$

in a steadily rotating frame of reference $\mathbf{x}^*$, where $\mathrm{tr}(\cdot)$ denotes the trace. For the problem of steady heat conduction,

$$\mathbf{v}^* = 0$$

and the field variables are time independent. Furthermore, as stated earlier, we are considering the case where an *extra body force* of the form of Eq. (18) is applied in the rotating frame so that

$$\mathbf{b}^* = \mathbf{b} + \mathbf{b}_E \tag{22}$$

By making use of these results, it follows that Eq. (19) is satisfied identically and Eqs. (20) and (21) take the form

$$\nabla^* \cdot \mathbf{T}^* + \rho^* \mathbf{b} = 0 \tag{23}$$

$$\nabla^* \cdot \mathbf{q}^* = 0 \tag{24}$$

If material frame-indifference is invoked, then the general constitutive Eqs. (5)–(7) must take the *same* form in the rotating frame $\mathbf{x}^*$, i.e.,

$$\begin{gathered}\mathbf{T}^*(\mathbf{x}, t) = \mathbf{T}[\mathbf{v}^*(\mathbf{x}', t'), \theta^*(\mathbf{x}', t'); \mathbf{x}, t] \\ \mathbf{x}' \in \mathscr{V}, \qquad t' \in (-\infty, t)\end{gathered} \tag{25}$$

$$\begin{gathered}\mathbf{q}^*(\mathbf{x}, t) = \mathbf{q}[\mathbf{v}^*(\mathbf{x}', t'), \theta^*(\mathbf{x}', t'); \mathbf{x}, t] \\ \mathbf{x}' \in \mathscr{V}, \qquad t' \in (-\infty, t)\end{gathered} \tag{26}$$

$$\begin{gathered}\varepsilon^*(\mathbf{x}, t) = \boldsymbol{\varepsilon}[\mathbf{v}^*(\mathbf{x}', t'), \theta^*(\mathbf{x}', t'); \mathbf{x}, t] \\ \mathbf{x}' \in \mathscr{V}, \qquad t' \in (-\infty, t)\end{gathered} \tag{27}$$

where, for the problem under consideration, $\mathbf{v}^*$ must be set equal to zero. It is clear that Eqs. (23) and (24), when solved in conjunction with Eqs. (25)–(27), have solutions such that

$$\mathbf{T}^* = \mathbf{T} \tag{28}$$

$$\mathbf{q}^* = \mathbf{q} \tag{29}$$

as a result of the fact that these equations are independent of the motion of the frame of reference. Thus, a continuum theory where material frame-indifference is invoked yields the *same values of stress and heat flux for the problem of steady heat conduction in a centrifugally balanced rotating gas no matter what its rotation rate* $\boldsymbol{\Omega}$ *is.*

It is now demonstrated that these results from continuum mechanics for heat conduction in a rigidly rotating gas are not generally consistent with the Boltzmann equation. In a steadily rotating frame $\mathbf{x}^*$, the Boltzmann equation takes the form [1]

$$\frac{\partial f^*}{\partial t^*} + \mathbf{c}^* \cdot \nabla^* f^* + [\mathbf{b}^* - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}^*) - 2\boldsymbol{\Omega} \times \mathbf{c}^*] \cdot \frac{\partial f^*}{\partial \mathbf{c}^*} = \mathbb{C}[f^*] \tag{30}$$

and, hence, is not of the same form as its inertial counterpart, Eq. (8). For the problem of steady heat conduction in a centrifugally balanced rotating gas, Eq. (30) reduces to the form

$$\frac{\partial f^*}{\partial t^*} + \mathbf{c}^* \cdot \nabla^* f^* + (\mathbf{b} - 2\mathbf{\Omega} \times \mathbf{c}^*) \cdot \frac{\partial f^*}{\partial \mathbf{c}^*} = \mathbb{C}[f^*] \tag{31}$$

where Eq. (22) is made use of. Obviously, the form of this equation changes depending on what value the rotation rate of the frame (i.e., the gas) takes. Hence, Eq. (31) will have solutions such that, in general,

$$f^* \neq f \tag{32}$$

and (since $\mathbf{v}^* = 0$)

$$\mathbf{T}^* = -m \int_{S_c} \mathbf{c}^* \mathbf{c}^* f^* d^3 c^* \neq \mathbf{T} \tag{33}$$

$$\mathbf{q}^* = \frac{m}{2} \int_{S_c} (\mathbf{c}^* \cdot \mathbf{c}^*) \, \mathbf{c}^* f^* d^3 c^* \neq \mathbf{q} \tag{34}$$

The values of the stress and heat flux for this test problem of heat conduction in a rigidly rotating gas depend, in general, on the rotation rate of the gas $\mathbf{\Omega}$. This is in contradiction of a continuum theory where material frame-indifference is invoked. A simple examination of Eq. (31) shows that this difference arises as a result of an unbalanced molecular Coriolis force—the same source that Müller [1] attributed to the failure of material frame-indifference of results obtained from the Maxwellian iteration.

It is now clear that the Boltzmann equation in its full generality is not consistent with material frame-indifference. However, it will be shown that this inconsistency is of no consequence unless the gas is *highly rarefied* and, thus, outside of the domain of classical continuum mechanics. By the introduction of the length scale l_0 and time scale t_0, which are, respectively, the mean free path and mean free time, Eq. (31) takes the dimensionless form

$$\frac{\partial f^*}{\partial t^*} + \mathbf{c}^* \cdot \nabla^* f^* + (\mathbf{b} - 2\Omega t_0 \boldsymbol{\lambda} \times \mathbf{c}^*) \cdot \frac{\partial f^*}{\partial \mathbf{c}^*} = \mathbb{C}[f^*] \tag{35}$$

where $\mathbf{\Omega} = \Omega \boldsymbol{\lambda}$ (i.e., $\boldsymbol{\lambda}$ is a unit vector along the axis of rotation). It is quite clear that if

$$\Omega t_0 \ll 1 \tag{36}$$

then the Coriolis term in Eq. (31) will be negligibly small in comparison to the convective term $\partial f^*/\partial t^* + \mathbf{c}^* \cdot \nabla^* f^*$, and hence, Eq. (31) will have solutions such that

$$f^* \approx f \tag{37}$$

$$\mathbf{T}^* \approx \mathbf{T} \tag{38}$$

$$\mathbf{q}^* \approx \mathbf{q} \tag{39}$$

Material frame-indifference would then be valid in a strong approximate sense.

The largest rotation rate that can be obtained in a feasible centrifuge is of the order of

$$\Omega = 10^4 \text{ s}^{-1} \tag{40}$$

which constitutes an enormously rapid rotation. Furthermore, for most common monatomic gases (e.g., hydrogen) at standard temperature and pressure, the mean free time is of the order of [14]

$$t_0 = 10^{-10} \text{ s} \tag{41}$$

Hence, the dimensionless rotation rate will be bounded by

$$\Omega t_0 \leqslant 10^{-6} \tag{42}$$

for gases which are strongly within the continuum limit. Hence, the error introduced by the application of material frame-indifference would be of the order of one part in a million or less! For the overwhelming majority of engineering applications, rotation rates are not encountered that are greater than 10^2 s^{-1}. In such circumstances, the mean free time would have to be such that

$$t_0 \geqslant 10^{-4} \text{ s} \tag{43}$$

in order for frame-dependent effects to be of real significance (i.e., to constitute more than a 1% effect). Of course, gases with such mean free times are *highly rarefied* and outside of the usual continuum description [14]. To be more specific, while the Boltzmann equation would still apply, constitutive-like equations obtained from the Chapman–Enskog and Maxwellian iterations would no longer be valid.

Finally, some comments should be made concerning how these results compare with the more recent study of Heckl and Müller [15] dealing with mixtures of gases. In this study (which was not based on a direct analysis of the Boltzmann equation) they concluded that, for a rigidly rotating gas, the frame-dependent terms obtained from the closure of the equations of transfer are of the order of the ratio of the time of free flight of a molecule to the period of rotation—a quantity which is extremely small unless the gas is highly dilute. Hence, the results obtained in this paper are consistent with those obtained by Heckl and Müller [15].

4. CONCLUSION

It has been demonstrated that the Boltzmann equation, in its full generality, is not consistent with the main physical consequence of material frame-indifference, which requires that the *values* of the stress tensor and heat flux vector remain unaffected when a rigid body motion is superimposed on a given thermomechanical process. This inconsistency, which arises because of the presence of unbalanced molecular Coriolis forces, was explicitly shown for the test problem of steady heat conduction in a centrifugally balanced rotating gas. For most common gases at standard temperature and pressure, the frame-dependent terms were shown to be negligibly small (i.e., to yield a correction of no more than one part in a million) even for the most rapid rotation which can be feasibly produced in the laboratory. However, for *highly rarefied* gases, the frame-dependent terms could make a nonnegligible contribution, thus making the thermophysical properties of such a gas vary depending on whether the frame of reference is inertial or not.

Classical continuum mechanics assumes the existence of constitutive equations where the stress and heat flux depend *locally* on the motion and temperature of the medium. For such equations to exist, the time scale of the molecular motion (e.g., the mean free time in an ideal gas) must be *extremely* small—the same limit in which the kinetic theory of gases was shown to be consistent with material frame-indifference. Hence, the results of this study strongly suggest that material frame-indifference can be invoked as an axiom in the formulation of such constitutive equations without introducing any significant error. In fact, since material frame-indifference provides such a powerful tool for restricting the allowable form of constitutive equations, it would simply be unintelligent not to make use of it in the formulation of phenomenological models, which are not expected to have an extraordinarily fine level of accuracy so that such inconsistencies would matter. However, there is a need for caution in the application of material frame-indifference in the formulation of some of the more modern continuum theories which include large-scale nonlocal effects [16, 17]. Such theories contain a microscale which can be significant in comparison to the geometrical scale of the problem. This constitutes a situation analogous to a highly rarefied gas, for which the results of this study suggest that the application of material frame-indifference could give rise to a nonnegligible error. Greater insight on this issue would be gained if an analytical or computational solution of the Boltzmann equation could be obtained for some simplified problem involving the rigid rotation of a highly rarefied gas.

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